Microscopic dynamics underlying anomalous diffusion

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The time-dependent Tsallis statistical distribution describing anomalous diffusion is usually obtained in the literature as the solution of a nonlinear Fokker-Planck (FP) equation [A.R. Plastino and A. Plastino, Physica A **222**, 347 (1995)]. The scope of the present paper is twofold. First, we show that this distribution can be obtained also as a solution of the nonlinear porous media equation. Second, we prove that the time-dependent Tsallis distribution can be obtained also as a solution of a linear FP equation [G. Kaniadakis and P. Quarati, Physica A **237**, 229 (1997)] with coefficients depending on the velocity, which describes a generalized Brownian motion. This linear FP equation is shown to arise from a microscopic dynamics governed by a standard Langevin equation in the presence of multiplicative noise.

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I. INTRODUCTION

Recently, Tsallis thermostatistics has received considerable attention in light of its growing application to a variety of physical systems [1]. The research has focused both on fundamental and phenomenological aspects of the issue [2].

Particular attention has been devoted to the issue of anomalous diffusion, where a significant amount of experimental evidence has been gathered (see Ref. [3] for a detailed bibliography). The description of a diffusive process (either classic or anomalous) is performed generally by adopting a time-dependent formalism. The Tsallis distribution, namely

$$p(v)=\frac{1}{Z_q}[1-(1-q)\beta v^2]^{1/(1-q)},$$

with $Z_q=\int_{\mathcal{R}}dv[1-(1-q)\beta v^2]^{1/(1-q)}$, has been first derived starting from the generalized entropy

$$S_q=\frac{1}{q-1}\left[1-\int_{\mathcal{R}}dv\, p^q\right],$$

using the maximum entropy principle under the constraint of conservation of particle number and energy, by solving the variational problem: $\delta(S_q-\beta E-\alpha N)=0$.

Similarly to the classic Boltzmann distribution, the Tsallis distribution can be also obtained as the steady-state distribution of a time-dependent Fokker-Planck (FP) equation. Recently, research on the derivation of the Tsallis distribution from FP equations has produced considerable results [3–13]. The research in this area can be classified in one of two classes.

First, linear FP equations are considered with diffusion and drift coefficients depending on the velocity. The dependence is chosen to lead to the Tsallis distribution as the equilibrium solution of the FP equation. Within the linear approach, two different choices of the drift and diffusion coefficients have been proposed. Stariolo [4] chooses a constant diffusion coefficient and alters the drift coefficient to include a generalized potential depending on the Tsallis parameter q. This approach introduces a more relevant modification of the classic Brownian approach. In Ref. [5], instead, the classic Brownian drift coefficient has been considered, but with a modified diffusion coefficient to include a quadratic velocity dependence. The two linear approaches described above are in reality just examples of an infinite class of linear FP models that give a Tsallis equilibrium distribution [6]. Clearly, the selection of a specific linear model among the class requires the introduction of other criteria beyond the simple requirement of leading to an equilibrium Tsallis distribution.

Second, nonlinear FP have been shown to lead to equilibrium Tsallis distributions. This approach, introduced by Plastino and Plastino [7] and continued by various authors [3,8–13], introduces a diffusion coefficient depending on powers of the distribution function. The drift, instead, can be equal to zero or described as in the classic Brownian motion. This latter approach, besides its elegance and simplicity, admits time-dependent solutions characterized by retaining at every time the form of a Tsallis distribution. This self-similarity of the evolution represent an important property of the nonlinear approach.

The present paper deals with the question of whether the linear [5] and nonlinear [7] FP approaches to the derivation of the Tsallis distributions are equivalent. The answer proven here is that indeed the two approaches are equivalent, in the sense that they both allow the presence of self-similar transients where the system is characterized by the Tsallis distribution at every instant.

In order to explain the microscopic origin of the anomalous diffusion associated with the nonlinear FP equation of Ref. [7], Borland suggested feedback from the macroscopic level to the microscopic one [13]. In the present work, we show that the nonlinear FP equation of Ref. [7] as well as the well-known nonlinear porous media equation, considered re-

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cently in the frame of Tsallis thermostatistics in Ref. [8], can be recast in the equivalent linear FP equation of Ref. [5]. This important result allows a deeper interpretation of the nonlinear FP equation describing the anomalous diffusion in terms of a linear Langevin microdynamics in the presence of a multiplicative noise.

The present work is organized as follows. In Sec. II, a generalized Brownian (GB) motion is derived from the Langevin equation in the presence of multiplicative noise. In Sec. III, the GB motion is shown to lead to a macroscopic motion described by the linear FP equation of Ref. [5] that admits as a solution a class of time-dependent Tsallis statistical distributions. In Secs. IV and V, the same distributions are shown to represent states governed also by the nonlinear FP equation of Ref. [7] and by the nonlinear porous media equation, respectively. Finally, in Sec. VI conclusions are drawn.

II. GENERALIZED BROWNIAN MOTION

We consider the microscopic process described by the following Langevin equation:

$$\frac{dv(t)}{dt}+h(t,v)=g(t,v)\Gamma(t), \tag{1}$$

with

$$\langle\Gamma(t)\rangle=0, \tag{2}$$

$$\langle\Gamma(t)\Gamma(t')\rangle=2\delta(t-t'). \tag{3}$$

The quantity $-mh(t,v)$ is the deterministic force acting on a particle of mass m and velocity $v(t)$ while $mg(t,v)\Gamma(t)$ is a stochastic force acting on the particle, with $\Gamma(t)$ a Gaussian random variable with zero mean and δ-correlation function. The presence of $g(t,v)$ in Eq. (1) implies that the particle is subject to a multiplicative noise. The distinction between additive [when $g(t,v)=\text{const}$] and multiplicative noise [when $g(t,v)\neq\text{const}$] is very significant when $g(t,v)$ is a time-dependent function. In this case, the question naturally arises related to the definition of the stochastic integral (Ito or Stratonovich definition). For a more detailed discussion on multiplicative noise, see Ref. [14]. The microscopic process described by Eq. (1) implies a macroscopic process described by the following linear FP equation:

$$\frac{\partial p(t,v)}{\partial t}=\frac{\partial}{\partial v}\left\{\left[J(t,v)+\frac{\partial D(t,v)}{\partial v}\right]p(t,v)+D(t,v)\frac{\partial p(t,v)}{\partial v}\right\}, \tag{4}$$

where the drift coefficient $J(t,v)$ and the diffusion coefficient $D(t,v)$ have the following expression:

$$J(t,v)=h(t,v), \tag{5}$$

$$D(t,v)=g(t,v)^2, \tag{6}$$

obtained using the Ito definition for the stochastic integral. Note that, for Brownian motion,

$$J(t,v)=\gamma(t)v, \tag{7}$$

$$D(t,v)=c(t), \tag{8}$$

the drift current in Eq. (4),

$$j_{\text{drift}}=\left[J(t,v)+\frac{\partial D(t,v)}{\partial v}\right]p(t,v), \tag{9}$$

is simplified,

$$j_{\text{drift}}=\gamma(t)v\,p(t,v), \tag{10}$$

and the current velocity j_{drift}/P becomes simply proportional to the viscous force $-mh(t,v)=-m\gamma(t)v$ of the microscopic process.

A problem arises in conjunction with the results just obtained, namely whether other motions, besides the Brownian motion, are characterized by a current velocity proportional to the viscous force. This issue corresponds to the existence of other solutions of the following equation for the unknown functions $D(t,v)$ and $J(t,v)$:

$$J(t,v)+\frac{\partial D(t,v)}{\partial v}=\theta(t)J(t,v), \tag{11}$$

in addition to the solution (7) and (8), relative to the Brownian motion. The issue is easily resolved and other solutions can be found. The more general solution is formed by copies of functions $J(t,v)$ and $D(t,v)$, where $D(t,v)$ is given by

$$D(t,v)=c(t)+[\theta(t)-1]\int J(v)dv, \tag{12}$$

while $J(t,v)$ remains arbitrary. The simplest solution, for which $J(t,v)$ is given by Eq. (7), provides the definition for a new generalized Brownian (GB) motion [5].

III. LINEAR FOKKER-PLANCK EQUATION

We consider the FP equation (4) for the GB processes. With the introduction of the dimensionless time τ:

$$d\tau=\theta(t)\gamma(t)dt, \tag{13}$$

and the functions $D(\tau)$, $\beta(\tau)$, and parameter q,

$$D(\tau)=\frac{c(t)}{\theta(t)\gamma(t)}, \tag{14}$$

$$(1-q)\beta(\tau)=\frac{1-\theta(t)}{2c(t)}, \tag{15}$$

the diffusion coefficient (12) with drift coefficient given by Eq. (7) can be written in the following form:

$$D(\tau,v)=D(\tau)[1-(1-q)\beta(\tau)v^2], \tag{16}$$

while after taking into account Eq. (7), the FP equation (4) becomes [5]:

$$\frac{\partial p(\tau,v)}{\partial \tau}=\frac{\partial}{\partial v}\left\{ v p(\tau,v)+D(\tau) \times[1-(1-q)\beta(\tau)v^2]\frac{\partial p(\tau,v)}{\partial v}\right\}. \tag{17}$$

The time-dependent solutions of Eq. (17) are sought using the following ansatz:

$$p(\tau,v)=\frac{1}{Z_q(\tau)}[1-(1-q)\beta(\tau)v^2]^{1/(1-q)}. \tag{18}$$

The above ansatz requires the solution to conserve at every time the form of a Tsallis distribution with time-dependent parameters Z_q and β. The time dependence of the two parameters determines the actual solution and is obtained easily substituting ansatz (18) in Eq. (17). It follows that the equations determining the evolution of $Z_q(\tau)$ and $\beta(\tau)$ are identical to the equations for the Brownian motion:

$$\frac{Z_q(\tau)}{Z_q(0)}=\left[\frac{\beta(0)}{\beta(\tau)}\right]^{1/2}, \tag{19}$$

$$\frac{d\beta(\tau)}{d\tau}=2\beta(\tau)-4D(\tau)\beta(\tau)^2. \tag{20}$$

The result above justifies the use of the term generalized Brownian motion to name the process defined by Eqs. (7) and (12). From Eq. (20) the condition below follows:

$$2\beta(\infty)D(\infty)=1, \tag{21}$$

again in complete similarity with Browinan motion. Equation (20) is solved easily with the substitution $y=\beta^{-1}$ that linearizes the equation

$$\beta(\tau)=\beta(\infty)\left\{1+\left[\frac{\beta(\infty)}{\beta(0)}-1+a(\tau)\right]\exp(-2\tau)\right\}^{-1}, \tag{22}$$

with

$$a(\tau)=2\int_0^{\tau}\left[\frac{D(\tau)}{D(\infty)}-1\right]\exp(-2\tau)d\tau. \tag{23}$$

From Eq. (19) it follows that

$$Z_q(\tau)\beta(\tau)^{1/2}=Z_q(0)\beta(0)^{1/2}=N_q. \tag{24}$$

The constant N_q is determined starting from the expression of $Z_q(\tau)$ given by

$$Z_q(\tau)=\int_{-\infty}^{+\infty}dv[1-(1-q)\beta(\tau)v^2]^{1/(1-q)}. \tag{25}$$

For $q\geqslant 1$ [15], it results

$$N_q=\frac{q+1}{2}\sqrt{\frac{q-1}{\pi}}\frac{\Gamma(1/2+1/(q-1))}{\Gamma(1/(q-1))}. \tag{26}$$

The final solution of Eq. (17) has the form

$$p(\tau,v)=N_q\beta(\tau)^{1/2}[1-(1-q)\beta(\tau)v^2]^{1/(1-q)}, \tag{27}$$

where $\beta(\tau)$ is given by Eqs. (22) and (23).

IV. NONLINEAR FOKKER-PLANCK EQUATION

The scope of the present and the next sections is to show that the time-dependent solution (27) obtained here of the linear FP equation (17) proposed in [5] is also a solution of nonlinear FP equations which can be obtained from the linear FP (17). The goal of the present section is to investigate the relationship between the linear FP (17) and the nonlinear FP equation proposed by Plastino and Plastino [7].

We start the proof by noting that Eq. (27) allows us to write

$$1-(1-q)\beta(\tau)v^2=N_q^{q-1}\beta(\tau)^{(q-1)/2}p(\tau,v)^{1-q}. \tag{28}$$

Besides, the following time-dependent function is defined:

$$D_1(\tau)=\frac{N_q^{q-1}}{2-q}D(\tau)\beta(\tau)^{(q-1)/2}. \tag{29}$$

Then, it follows that Eq. (17) can be rewritten as

$$\frac{\partial p(\tau,v)}{\partial \tau}=\frac{\partial}{\partial v}\left\{v p(\tau,v)+D_1(\tau)\frac{\partial}{\partial v}[p(\tau,v)]^{2-q}\right\}, \tag{30}$$

which is identical to the equation proposed by Plastino and Plastino that was solved using the same ansatz (18) used above but assuming that $D_1(\tau)$ is constant.

The procedure outlined in the preceding section leads to the same relationship (19) between $Z_q(\tau)$ and $\beta(\tau)$, and $\beta(\tau)$ is governed by the following evolution equation:

$$\frac{d\beta(\tau)}{d\tau}=2\beta(\tau)-2\beta(\infty)^{(q-3)/2}\frac{D_1(\tau)}{D_1(\infty)}\beta(\tau)^{(5-q)/2}. \tag{31}$$

The condition (21) transforms now as

$$2\beta(\infty)^{(3-q)/2}D_1(\infty)=\frac{N_q^{q-1}}{2-q}. \tag{32}$$

As above, a transformation $y=\beta^{(q-3)/2}$ linearizes Eq. (31) and the general solution follows easily:

$$\beta(\tau)=\beta(\infty)\left(1+\left\{\left[\frac{\beta(\infty)}{\beta(0)}\right]^{(3-q)/2}-1+b_q(\tau)\right\} \times\exp[(q-3)\tau]\right)^{2/(q-3)}, \tag{33}$$

$$b_q(\tau)=(3-q)\int_0^{[(3-q)\tau]/2}\left[\frac{D_1(\tau)}{D_1(\infty)}-1\right]\exp[(q-3)\tau]d\tau. \tag{34}$$

The complete solution of Eq. (30) is given by Eq. (27), where now $\beta(\tau)$ is expressed as a function of $D_1(\tau)$ by Eqs. (33) and (34). In the special case of $D_1(\tau)$ constant, the results presented in the literature [3,7] are recovered.

V. NONLINEAR POROUS MEDIA EQUATION

For the solutions (27) considered above, the following result follows readily:

$$v p(\tau,v) = \frac{N_q^{q-1}}{2(q-2)} \beta(\tau)^{(q-3)/2} \frac{\partial}{\partial v} [p(\tau,v)]^{2-q}. \quad (35)$$

Substituting the relationship above in Eq. (30), it follows that

$$\frac{\partial p(\tau,v)}{\partial \tau} = D_2(\tau) \frac{\partial^2}{\partial v^2} [p(\tau,v)]^{2-q}, \quad (36)$$

where

$$D_2(\tau) = \frac{N_q^{q-1}}{q-2} \beta(\tau)^{(q-3)/2} \left[\frac{1}{2} - \beta(\tau) D(\tau) \right]. \quad (37)$$

As a consequence, Eq. (36) is the well known nonlinear porous media equation, widely used in condensed-matter physics and considered recently in Ref. [8]. The current of particles is given by

$$j(\tau,v) = -D_2(\tau) \frac{\partial}{\partial v} [p(\tau,v)]^{2-q}, \quad (38)$$

which generalizes the Fick law (indeed, for $q=1$ the classic Fick law is recovered).

Clearly, the time-dependent Tsallis distribution (18) is a solution of Eq. (36) as well. $Z_q(\tau)$ and $\beta(\tau)$ are connected by Eq. (19) and the solution acquires the form (27) while the evolution law for $\beta(\tau)$ follows from $D_2(\tau)$. The final differential equation is

$$\frac{d\beta(\tau)}{d\tau} = 4(q-2) N_q^{1-q} D_2(\tau) \beta(\tau)^{(5-q)/2}, \quad (39)$$

which is solved readily to obtain

$$\beta(\tau) = \beta(0) \left[1 + 2(q-2)(q-3) N_q^{1-q} \times \beta(0)^{(3-q)/2} \int_0^\tau D_2(\tau) d\tau \right]^{2/(q-3)}. \quad (40)$$

From Eqs. (37) and (39) it follows that $D_2(\infty)=0$, a condition that guarantees that the current (38) vanishes for $\tau \to \infty$, as needed to obtain an equilibrium state.

VI. CONCLUSIONS

In the present work, the same time-dependent Tsallis statistical distribution given in Eq. (27) is a solution of all three equations considered, Eq. (17), Eq. (30), and Eq. (36), and describes anomalous diffusion. The time evolution of $\beta(\tau)$ is the same in all the cases considered and can be expressed in terms of the three functions $D(\tau)$, $D_1(\tau)$, $D_2(\tau)$ that are inter-related.

The relationships derived above allow us to interpret the states (27) at a microscopic level. Such states are solutions of nonlinear FP equations [Eqs. (30) and (36)] and consequently describe anomalous diffusion (when $q \neq 1$). But the results obtained above prove that anomalous diffusion can also be described with the linear FP equation (17), which has a variable diffusion coefficient given by Eq. (16).

The primary result of this equivalence is that the linear FP equation (17) can be related directly to the microscopic dynamic model expressed by the Langevin equation (1) linking in this way macroscopic processes described by anomalous diffusion with microscopic processes characterized by multiplicative noise.

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